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A THERMO-KINETIC VIEW OF ELASTIC STABILITY THEORY

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Abstract—It seems to be common to regard thermodynamic stability and mechanical stability as two distinct subjects. We here explore the possibility of combining the two in a conceptually clear manner, in a rather limited context. Primarily, we work within the contexts of nonlinear elasticity and thermoelasticity theories, exploring relations between energy criteria for stability and consequences of a kinetic definition of stability.

1. PRELIMINARIES

AS A BASIS for our study, we first summarize common attributes of the more conventional continuum theories, including those mentioned above. In Cartesian tensor notation, there are the laws of conservation or balance of mass, linear momentum and momentum, which can be written as

$$\int \rho \, \mathrm{d}V \Big|_{t_1}^{t_2} = 0, \tag{1}$$

$$\int \rho \dot{x}_i \, \mathrm{d}V \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathrm{d}t \left(\oint T_{i\alpha} \, \mathrm{d}S_\alpha + \int \rho f_i \, \mathrm{d}V \right), \tag{2}$$

$$\int \rho x_{[i} \dot{x}_{j]} \, \mathrm{d}V \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathrm{d}t \left(\oint x_{[i} T_{j]\alpha} \, \mathrm{d}S_\alpha + \int \rho x_{[i} f_{j]} \mathrm{d}V \right), \tag{3}$$

t_1 and $t_2 \geq t_1$ being any two times. The integrands are considered as functions of time t and material coordinates X_α, interpretable as coordinates of particles in a convenient reference configuration. For a given body, the reference configuration is a region R, independent of time, the region of integration in (1)–(3). On its boundary ∂R, $\mathrm{d}S_\alpha$ denotes the outward directed vector element of area. Further, ρ is the mass per unit reference volume, x_i the present coordinates of a particle, $\dot{x}_i$ its velocity, and f_i the body force per unit mass. The tensor $T_{i\alpha}$ is the Piola–Kirchhoff stress, sometimes called engineering stress. Finally, in (3), square brackets denote "the antisymmetric part of". These integral forms apply to most physically acceptable solutions, including what are commonly called stress waves and shock waves.

Particularly in the theories covering thermal effects and to some extent in the rest, we use the energy equation

$$\int \rho(\varepsilon + \tfrac{1}{2}\dot{x}_i \dot{x}_i) \, \mathrm{d}V \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathrm{d}t \Big[\oint (T_{i\alpha} \dot{x}_i - Q_\alpha) \, \mathrm{d}S_\alpha + \int \rho f_i \dot{x}_i \, \mathrm{d}V \Big], \tag{4}$$

where ε is internal energy per unit mass, Q_α the heat flux vector, reckoned per unit area in the reference configuration. We exclude the volume sources of heat sometimes included. Also, there is the Clausius–Duhem inequality

$$\int \rho \eta \, \mathrm{d}V \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathrm{d}t \oint \frac{Q_\alpha}{T} \mathrm{d}S_\alpha \geq 0, \tag{5}$$

η being entropy per unit mass, T absolute temperature. Thermoelasticity theories here considered are designed so that this inequality always holds for reasonably smooth solutions, though shock waves might sometimes be exceptional.

To be definite, we concentrate on problems roughly corresponding to what is generally called stability under dead loading in elastic stability theory. Various other cases follow a similar pattern.* Consider a theory for which (1)–(5) apply rigorously. Consider a time independent solution, hereafter called the *rest solution*, defined in a region R, with

$$T = \hat{T} = \text{const.} > 0, \qquad f_i = \hat{f}_i \text{ in } R, \tag{6}$$

$$T_{i\alpha}\,\mathrm{d}S_\alpha = \hat{T}_i\,\mathrm{d}S \text{ on } \partial R. \tag{7}$$

Henceforth, "hats" always denote quantities associated with it. Consider any other solution $\mathscr{D}$, called a *disturbance*, defined in R for times in some interval $t_3 \le t \le t_1$. For $t > t_1$, we adjust the forces and surface temperature as follows:

$$T = \hat{T}, \qquad T_{i\alpha}\,\mathrm{d}S_\alpha = \hat{T}_i\,\mathrm{d}S \text{ on } \partial R, \qquad t > t_1 \tag{8}$$

$$f_i = \hat{f}_i \text{ in } R, \qquad t > t_1. \tag{9}$$

Assume there is at least one solution $\mathscr{S}$, called a *transient*, with $\mathscr{S} = \mathscr{D}$ for $t_3 \le t \le t_1$, satisfying (8) and (9) for $t > t_1$. For $\mathscr{S}$, we have from (4), (5), (8) and (9)

$$\int \rho(\varepsilon - \hat{T}\eta + \tfrac{1}{2}\dot{x}_i\dot{x}_i)\,\mathrm{d}V\Big|_{t_1}^{t_2} - \int \mathrm{d}t\left(\oint \hat{T}_i\dot{x}_i\,\mathrm{d}S + \int \rho\hat{f}_i\dot{x}_i\,\mathrm{d}V\right)$$
$$= -\hat{T}\left(\int \rho\eta\,\mathrm{d}V\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\mathrm{d}t\oint \frac{Q_\alpha}{T}\,\mathrm{d}S_\alpha\right) \le 0. \tag{10}$$

Now assume that, as $t \to \infty$, $\mathscr{S}$ converges to the rest solution. Bearing in mind that quantities bearing hats are independent of time, we obtain

$$\mathrm{X} \equiv \int \rho\varphi\,\mathrm{d}V\Big|_{t_1}^{\infty} - \int \rho[(T-\hat{T})\eta + \tfrac{1}{2}\dot{u}_i\dot{u}_i]\,\mathrm{d}V\Big|_{t=t_1} + \oint \hat{T}_i u_i\,\mathrm{d}S + \int \rho\hat{f}_i u_i\,\mathrm{d}V \le 0, \tag{11}$$

where

$$\varphi = \varepsilon - T\eta \tag{12}$$

is the Helmholtz free energy per unit mass and

$$u_i(X_\alpha, t) = x_i(X_\alpha, t) - \hat{x}_i(X_\alpha) \tag{13}$$

represents the displacement from the rest position. Here and in the following, we operate formally, assuming for example that

$$\lim_{t\to\infty}\int \rho\dot{x}_i\dot{x}_i\,\mathrm{d}V = \int \lim_{t\to\infty}\rho\dot{u}_i\dot{u}_i\,\mathrm{d}V = 0. \tag{14}$$

In various ways, one can make precise the statement that "$\mathscr{S}$ converges" so as to validate these operations. There are subtle points concerning the most appropriate definition which we choose not to investigate.

* A comprehensive survey of the general theory of elastic stability is given by Truesdell and Noll [1, §68].

It is of course conceivable that a given disturbance would admit no transient. If it does, and if every such continuation converges to the rest solution, we say that the rest solution is *kinetically stable with respect to the disturbance* $\mathscr{D}$. Equation (11) provides a necessary condition for this. It, or variations of it provide a basis for energy criteria used in elastic stability theory, and extensions of such criteria to more complex theories.

Strictly speaking, we never have kinetic stability with respect to all disturbances. From (2), (8) and (9) we can infer that, for a transient to converge to the rest solution,

$$\int \rho \dot{x}_i \, \mathrm{d}V \bigg|_{t=t_1} = \int \rho \dot{x}_i \, \mathrm{d}V \bigg|_{t>t_1} = \int \rho \dot{x}_i \, \mathrm{d}V \bigg|_{t\to\infty} = 0,$$

from which

$$\int \rho u_i \, \mathrm{d}V \bigg|_{t=t_1} = 0. \tag{15}$$

That is, at time t_1, the center of mass must be in its rest position. Intuitively, for a given material, it seems unlikely that a given rest solution will be stable with respect to all other disturbances; a sufficiently strong blow may well "destabilize" it. Practically, needs to be satisfied by a stability theory do vary, it being one thing to design a structure to withstand gentle breezes, quite another when it must survive hurricanes. Commonly, we select some set of disturbances, often a restricted set of infinitesimal disturbances, then seek to determine whether we have stability with respect to these. An alternative which might be more fruitful would involve seeking to characterize the set of disturbances with respect to which we do or do not have stability, leaving it to the designer to decide whether the destabilizing disturbances are likely to be encountered. We later comment on an approach to this problem which may be feasible in some cases.

Thus far, we have said rather little about the theories to be used. These could incorporate viscoelastic and thermal effects, at least as long as these fall within the framework discussed by Coleman [2]. For more detailed exploration, we turn to more special theories involving some simplifying features.

2. THERMOELASTIC STABILITY

We now turn to nonlinear thermoelasticity theory.* For simplicity, we neglect body forces. It would in fact be simpler, but unrealistic to assume they can be varied at will. We then have constitutive equations of the form

$$\varphi = \varphi(x_{i,\alpha}, T, X_\beta) = \varepsilon - T\eta, \tag{16}$$

$$\eta = -\partial\varphi/\partial T, \tag{17}$$

$$T_{i\alpha} = \rho \partial\varphi/\partial x_{i,\alpha}, \tag{18}$$

$$Q_\alpha = Q_\alpha(x_{k,\beta}, T, T_{,\gamma}, X_\delta), \tag{19}$$

$$Q_\alpha T_{,\alpha} \leq 0. \tag{20}$$

The inequality (20), a consequence of (5), implies that†

$$Q_\alpha = 0 \quad \text{when} \quad T_{,\beta} = 0. \tag{21}$$

* For a modern development of this, cf. Coleman and Noll [3].

† Cf. Truesdell and Noll [1, §96].

The governing differential equations are

$$T_{i\alpha,\alpha} = \rho \ddot{x}_i, \tag{22}$$

$$\rho T \dot{\eta} = -Q_{\alpha,\alpha}. \tag{23}$$

In (16) and (19), $x_{i,\alpha}$ can be replaced by any of the finite material strain measures, φ and Q_α being insensitive to rigid rotations.

In particular, for static deformations with $T = \hat{T} =$ const., such a theory reduces to a nonlinear elasticity theory for which W, the strain energy per unit reference volume, is given by

$$W = W(x_{i,\alpha}, X_\beta) = \rho\varphi(x_{i,\alpha}, \hat{T}, X_\beta). \tag{24}$$

In the present context, disturbances and transients will be thermoelastic, but not necessarily elastic solutions. Rest solutions will of course be elastostatic in the obvious sense.

As a general rule, these thermoelastic equations admit solutions to an initial value problem wherein we prescribe x_i, $\dot{x}_i$ and T as analytic functions of X_α at a given time t_1. To construct such a solution, we use the governing equations to calculate time derivatives of all orders of displacement and temperature, thereby obtaining a formal power series in the time for these functions of their arguments at values corresponding to the given data. Further, examination shows that we should have

$$T\, \partial\eta/\partial T \neq 0 \tag{25}$$

for the data given. We can then use the Cauchy–Kowalewski existence theorem to establish the fact that the formal power series converges to a solution, analytic in the time, for t sufficiently close to t_1. This makes it plausible to assume, as we shall, that there are disturbances which produce essentially arbitrary smooth values of x_i, $\dot{x}_i$ and T at a given instant. Clearly, we should avoid singularities in the constitutive equations and places where (25) fails. On physical grounds, the left side of (25) might be expected to be positive, being proportional to the specific heat at constant deformation. However, one should be alert to the possibility that exceptions to (25) may well occur for special choices of φ such as obtained from polynomial approximations and these may well imply some type of instability, perhaps merely indicating that the range of applicability of the theory is exceeded. Reverting to our assumption, we can calculate initial surface tractions and surface temperature. In principle, the analytic solution supplies such data as long as it exists. Ordinarily this will disagree with that which we desire, given by (8). It may still be possible to continue the disturbance in time, in a non-analytic fashion, so as to satisfy (8). If the initial data disagrees with (8), any transient necessarily involves a rather strong singularity, perhaps involving shock waves, generated by the abrupt change in boundary conditions. As a matter of choice we can include or exclude such disturbances. If we exclude them, we must restrict initial data so that they satisfy

$$\hat{T}_{i\alpha}\, dS_\alpha = \hat{T}_i\, dS, \qquad T = \hat{T} \text{ on } \partial R, \qquad t = t_1. \tag{26}$$

We do not expect that these restrictions will rule out strong singularities. Pragmatically, we do not know how to accomplish this.* It is rather clear that there are physically interesting instability phenomena correlating with this, scabbing and other fracture

* For various linearized theories, Shield [4] presents conditions on initial data which guarantee that solutions have definite continuity properties, granted that they exist.

phenomena being reasonably well understood in terms elementary linear elastic wave analyses. Said differently, possible breakdowns in existence of relatively smooth solutions are likely to be significant.* It does not seem feasible to here explore these existence theoretic questions further, so we take a different tack.

Clearly, a transient interrupted at any particular time, will serve as a disturbance. For stability with respect to the original disturbance, we must also have stability with respect to this disturbance, so (11) must apply to it. It seems worthwhile to explore to what extent (11) might be sufficient for stability. To this end, consider

$$\mathrm{X}(t) = \int \rho[\hat{\varphi} - \varphi - (T - \hat{T})\eta + \tfrac{1}{2}\dot{u}_i\dot{u}_i]\,\mathrm{d}V + \int \hat{T}_i u_i\,\mathrm{d}S \tag{27}$$

as a functional of time dependent vector fields, u_i and scalars $T - \hat{T}$. For this, we assume that the constitutive equations apply and that a fixed rest solution is given, but it is not essential that (22) and (23) be satisfied. The guiding principle is that u_i and $T - \hat{T}$ should be in a function space which includes disturbances and transients which we wish to consider. We can limit them by imposing conditions such as are suggested by (15) and (26), for example. Suppose that

$$\mathrm{X}(t) \leq 0 \tag{28}$$

for the functions considered admissible. The value of X then serves as a rough measure of the departure from the rest solution, being more reliable if, in (28), the equality holds only when there is no departure. For linearized theories, it can be reduced to a quadratic integral which, when strictly positive, provides a natural Hilbert space norm.

When we evaluate X for a thermoelastic solution satisfying (26), we have

$$\begin{aligned}
\dot{\mathrm{X}} &= -\int [T_{i\alpha}\dot{u}_{i,\alpha} - \rho\eta\,\dot{T} + \rho\,\dot{T}\eta + \rho(T - \hat{T})\dot{\eta} + \dot{u}_i T_{i\alpha,\alpha}]\,\mathrm{d}V + \oint T_{i\alpha}\dot{u}_i\,\mathrm{d}S_\alpha \\
&= -\int [T_{i\alpha}\dot{u}_{i,\alpha} + \rho(T - \hat{T})\dot{\eta} + \dot{u}_i T_{i\alpha,\alpha} - (T_{i\alpha}\dot{u}_i)_{,\alpha}]\,\mathrm{d}V \\
&= \int (1 - \hat{T}/T) Q_{\alpha,\alpha}\,\mathrm{d}V \\
&= \oint (1 - \hat{T}/T) Q_\alpha\,\mathrm{d}S_\alpha - \int (1 - \hat{T}/T)_{,\alpha} Q_\alpha\,\mathrm{d}V \\
&= -\,\hat{T}\int (Q_\alpha T_{,\alpha})/T^2\,\mathrm{d}V \geq 0
\end{aligned} \tag{29}$$

so X increases monotonically with time, generally being strictly increasing as long as T differs from its equilibrium value. As long as (28) applies, and to the extent that X is a satisfactory measure of departure from equilibrium, we thus tend to get closer to equilibrium. There is no guarantee that the rest solution is attained. The solution could fail to exist after a finite time, possibly go into steady state isothermal oscillation about the rest solution. If we have selected the function space unwisely, we may leave it and find that (28) no longer applies, etc. In any event (28) has some pertinence with respect to the stability problem. In particular, if the strict form of (28) holds, we easily see that if the material is initially in the rest configuration and we maintain the boundary conditions at rest values, the material must stay at rest. In linearized theories, this amounts to the

* It seems pertinent to note that the explanation of internal fracture of rubber proposed by Gent and Lindley [5] exploits the fact that a certain elastostatic problem has no solution if the load be too large.

statement that there is at most one solution of the initial-boundary value problem of the type here discussed.

We further explore (11), seeking to correlate it with an energy criterion for elastic stability. To this end, we introduce a finite Taylor expansion in the temperature writing

$$\begin{aligned}\rho[\varphi+(T-\hat{T})\eta] &= \rho[\varphi-(T-\hat{T})\partial\varphi/\partial T] \\ &= W+\tfrac{1}{2}K\rho(T-\hat{T})^2, \end{aligned} \tag{30}$$

where W is given by (29) and

$$\begin{aligned}K &= 2\rho(T-\hat{T})^{-2}[\varphi-(T-\hat{T})\partial\varphi/\partial T-W/\rho] \\ &= -\left[\frac{\partial^2\varphi}{\partial T^2}+(T-\hat{T})\frac{\partial^3\varphi}{\partial T^3}\right]^* \end{aligned} \tag{31}$$

the star denoting that this quantity is to be evaluated for some temperature between T and $\hat{T}$. Thus (27)–(28) become

$$X = \int(\hat{W}-W)\,dV + \oint \hat{T}_i u_i\,dS - \tfrac{1}{2}\int\rho[K(T-\hat{T})^2+\dot{u}_i\dot{u}_i]\,dV \leq 0. \tag{32}$$

The first two terms are independent of T and $\dot{u}_i$. Thus if this holds for a sufficiently broad set of fields to permit us to vary these independently of u_i, or if we may merely set $T = \hat{T}$, $\dot{u}_i = 0$ as a possibility, we must have

$$\int(\hat{W}-W)\,dV + \oint \hat{T}_i u_i\,dS \leq 0 \tag{33}$$

for all admissible displacements. Similarly, another condition is obtained by setting $u_i = \dot{u}_i = 0$ and granting that $T-\hat{T}$ can be varied arbitrarily, viz.

$$K\Big|_{u_i=0}(T-\hat{T})^2 = \rho[\varphi-(T-\hat{T})\partial\varphi/\partial T-W|_{u_i=0} \geq 0. \tag{34}$$

From (31), this clearly implies that

$$-\hat{T}\partial^2\hat{\varphi}/\partial T^2 = \hat{T}\partial\hat{\eta}/\partial T \geq 0. \tag{35}$$

That is, for the rest solution, the specific heat at constant deformation must be non-negative. For sufficiently small disturbances, (33) and (34) imply (32).

From what was said earlier concerning arbitrariness of data for the initial value problem, it is not entirely unreasonable to require (33) to hold for essentially arbitrary smooth displacement fields. Then (33) becomes the criterion for elastic stability under dead loading proposed by Pearson [6]. For reasons discussed by Beatty [7], this criterion is inappropriate for analyzing simple buckling problems; it is easily shown that, in cases of comprehensive loading, (32) necessarily fails when u_i describes a suitably selected rigid rotation, no matter how small the load. Experimentally, we must take pains to exclude this "misalignment" type of disturbance if we hope to measure a buckling load for a beam which is at all close to the Euler load. This provides a homely example of a case where we have stability with respect to some types of disturbance, not with respect to others. Beatty [7] proposes that, in such cases (33) be required to hold only for u_i which give rise to zero resultant moment,

$$\oint u_{[i}\,\hat{T}_{j]}\,dS = 0. \tag{36}$$

A comparable assumption in beam theory is that the ends of the beam are pinned. The work of Holden [8] indicates how one can use this to obtain safe estimates of critical loads in terms of Korn's constant, which depends only on the shape of the region. From this and Pearson's analysis, which is likely to give an unsafe estimate, it seems likely that, for beams of sufficiently large length to width ratio, the three-dimensional theory probably gives a load somewhat smaller but of the same order of magnitude as the Euler load. An alternative constraint, suggested in (26), is that we require that the surface tractions corresponding to u_i match those for the rest solution. This possibility is briefly discussed by Truesdell and Noll [1, § 68], but no predictions have been obtained from it. A third alternative obtains as follows: pick one of the necessary conditions for stability, as here defined. To be definite, pick (33). Use the left-hand side to divide all displacement fields into two sets, according as the value of this functional is positive or not. From what has been said, those which make it positive clearly represent disturbances with respect to which we do not have stability. For the remaining set we may have stability. It would of course be preferable to isolate those for which we definitely do have stability, but we lack useable criteria for this. In any event, this provides a basis for distinguishing at least some of the destabilizing disturbances. For estimating how the amplitude of a disturbance influences stability, we might proceed naively, writing

$$u_i = \varepsilon v_i \tag{37}$$

v_i being a fixed vector field, ε a parameter. Then the left side of (33) reduces to a function of ε. If it is negative for small ε, we can define a critical amplitude in the obvious manner. Since a small disturbance may produce a large transient, etc. such an estimate is trustworthy only from the point of view of establishing some destabilizing disturbances. However, I do not know of a completely satisfactory way of treating such problems.

There is some formal similarity between the inequality (11) and some proposed by Coleman and Noll [8], say their (15.6). Theirs have been considered more as restrictions on admissible constitutive equations, while we certainly envisage the possibility that ours are sometimes violated. I see no compelling physical reason to think that any such inequality holds universally. It is natural to expect that any such restriction will tend to exclude some instabilities which might otherwise be predicted. Various restrictions which have been considered are discussed by Truesdell and Noll [1, § 52].

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Résumé—Il semble habituel de considerer la stabilité thermodynamique et la stabilité mécanique comme deux sujets distincts. Nous explorons ici la possibilité de combiner ces deux sujets d'une manière absolument éclairée, dans un contexte limité. Principalement, nous travaillons avec les contextes de l'élasticité non-linéaire et les théories de thermoélasticité, explorant les relations entre les critères d'énergie de la stabilité et les conséquences d'une définition cinétique de stabilité.

Zusammenfassung—Es scheint allgemein üblich zu sein, Thermodynamische Stabilität und Mechanische Stabilität als zwei ausdrücklich verschiedene Subjekte zu betrachten. Wir untersuchen hier die Möglichkeit, die beiden in einem klaren Begriff eines ziemlich beschränkten Zusammenhanges zu verbinden. Vorwiegend, wir arbeiten innerhalb des Begriffes von nichtlinearer Elastizität und thermoelatischen Theorien in der Untersuchung von Beziehungen zwischen Energie Kriterium für Stabilität und Folgen einer kinetischen Erklärung der Stabilität.

Абстракт—Кажется обычным рассматривать термодинамическую устойчивость и механическую устойучивость, как две различных темы. Здесь мы исследуем возможность соединения того и другого схематически ясным способом в довольно ограниченной связи. Мы работаем, главным образом в пределах теорий нелинейной еластичности и термоеластичности, исследуя отношения между критерием енергии для устойчивости и следствиями кинетического определения устойчивости.